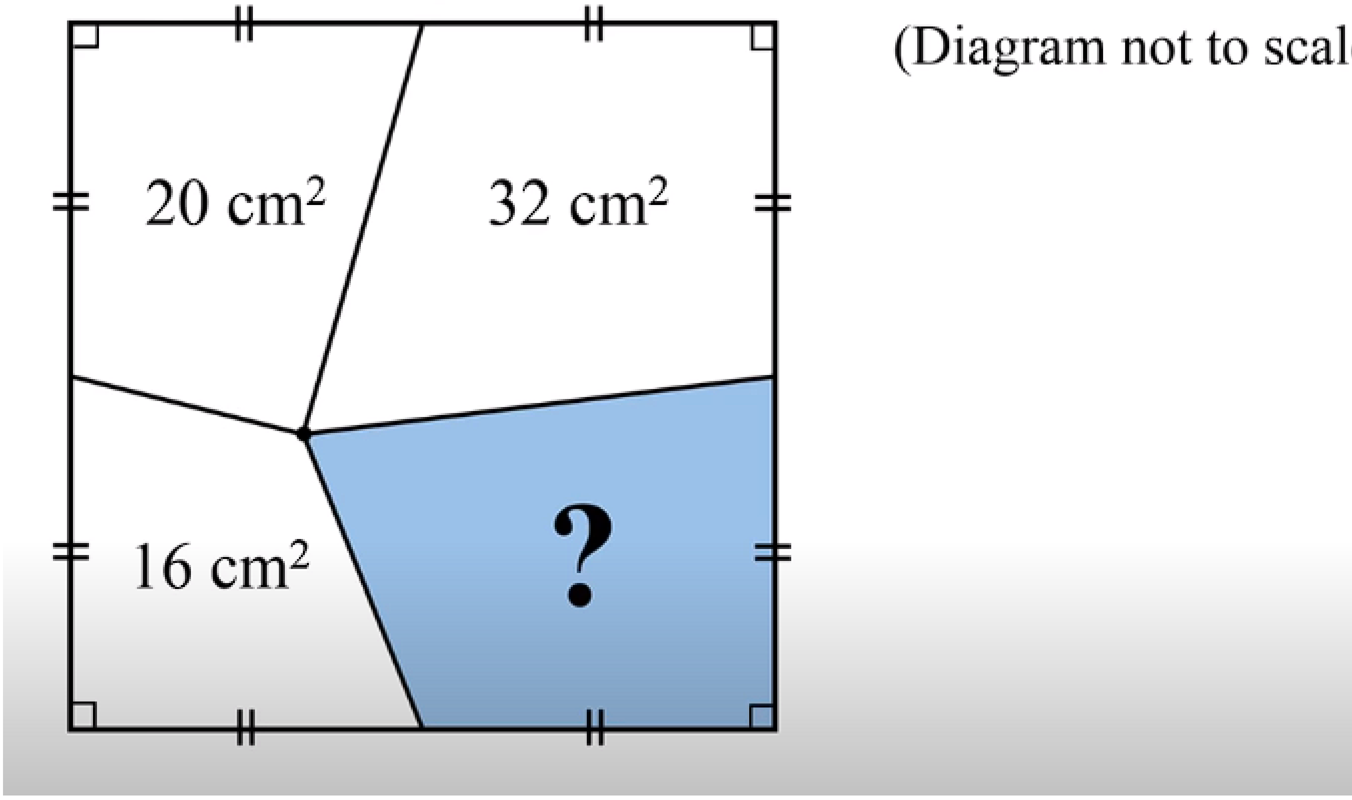
October Math Gems

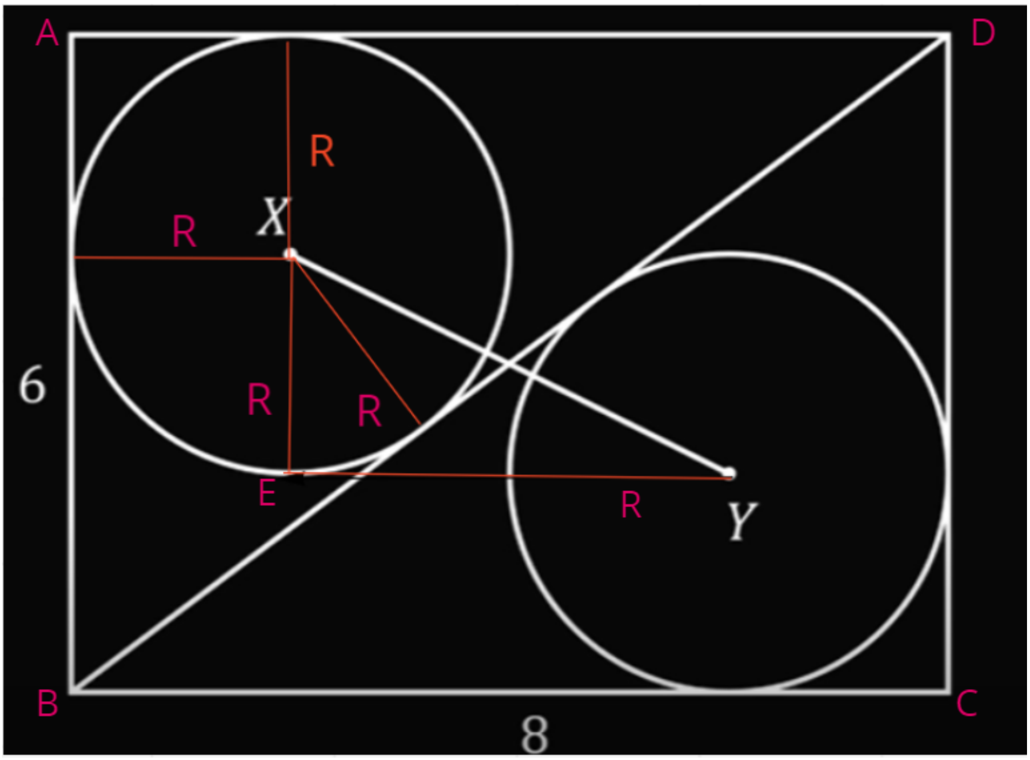
Problem of the week 26

§1 Problems

Problem 1.1. What is the length of $|XY|$



Solution. First, we can see that the length of $\overline{EY}$ is $8-2R$ and the length of $\overline{XE}$ is $6-2R$. Therefore the length of $\overline{XY}$ is equal to $\sqrt{(8-2R)^2+(6-2R)^2}$. Second, we can see that the area of triangle BCD is equal to 24, and the area of triangle ABC is equal to the areas of the triangles AXB, AXD, and XDB. Their areas are $3R$, $4R$, and $5R$. Therefore, $3R+4R+5R=24$. So, R is 2. We can substitute now in the first equation and get that the length of $\overline{XY}$ is $\sqrt{20}$.



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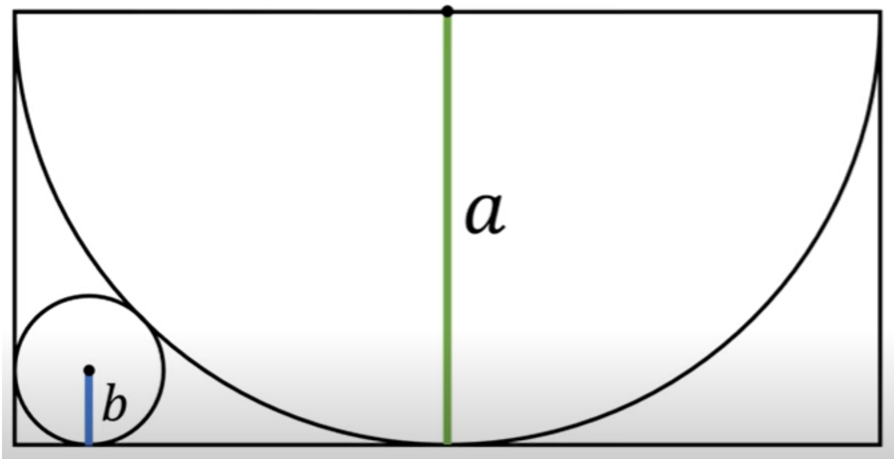
Problem 1.2. Suppose that α, β, and γ are the roots of the equation $x^3+3x^2-24x+1=0$. Find the value $\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}$

Solution. First, we need to try to factorize the equation. So, we can add $27x$ to both sides, and factorize it, which gives us the equation $(x+1)^3 = 27x$. Taking the cubic root to both sides we get that $\frac{x+1}{3} = \sqrt[3]{x}$. Since α, β, and γ are roots of the original equation, then they satisfy the equation we just got. Therefore, $\frac{\alpha+1}{3} = \sqrt[3]{\alpha}$, $\frac{\beta+1}{3} = \sqrt[3]{\beta}$, and $\frac{\gamma+1}{3} = \sqrt[3]{\gamma}$. Summing these all up, we get $\frac{\alpha+\beta+\gamma+3}{3} = \sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma}$. Using the fact that the sum of the roots of the cubic equation is $\frac{-b}{a}$, we see that $\alpha+\beta+\gamma = \frac{-3}{1} = -3$. Therefore, $\sqrt[3]{\alpha}+\sqrt[3]{\beta}+\sqrt[3]{\gamma} = \frac{-3+3}{3} = 0$ □

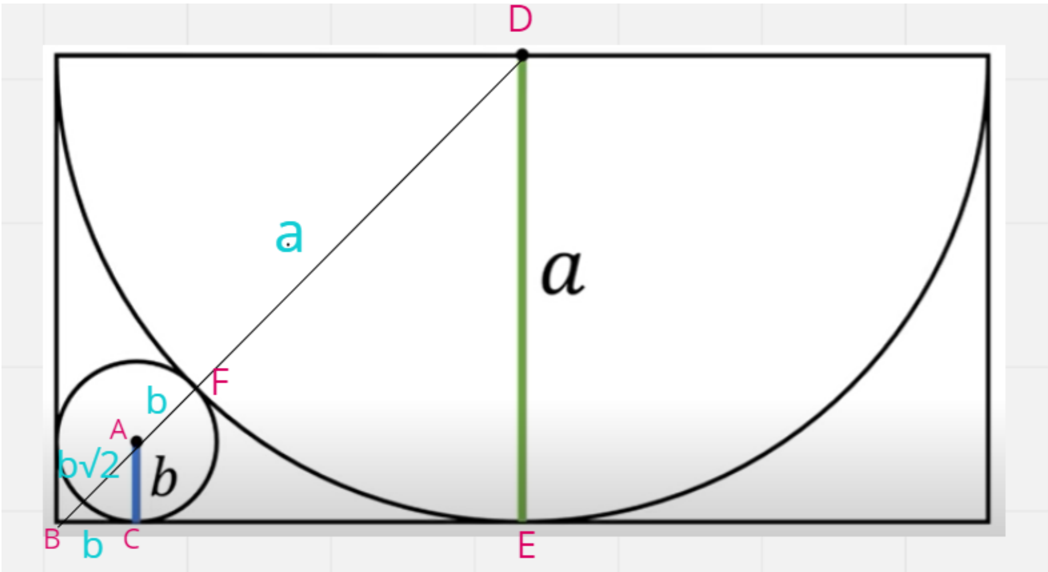
Problem 1.3. Find all real solution for the eqaution $\sqrt[3]{x}+\sqrt[3]{x-16} = \sqrt[3]{x-8}$

Solution. First, we could try to cube both sides of the equation. So, we get $x+(x-16)+3\sqrt[3]{x}\sqrt[3]{x-16}\sqrt[3]{x-8} = x-8$. Simplifying the equation, we get $-(x-8) = 3\sqrt[3]{x}\sqrt[3]{x-16}\sqrt[3]{x-8}$. Cubing both sides and moving $-(x-8)$ to the other side and taking a common factor of $(x-8)$, we get $(x-8)(27(x-16)+(x-8)^2) = 0$. After simplifying the polynomial within the brackets, we get that $4(x-8)(7x^2-112x+16) = 0$. Therefore, the possible solutions for x are $(8, 8 \pm \frac{12\sqrt{21}}{7})$ □

Problem 1.4. Find $\frac{a}{b}$



Solution. From the image below, we can see that the length of $\overline{DB}$ is equal to $a\sqrt{2}$. Also, we can deduce that $a\sqrt{2} = a + b + b\sqrt{2}$. Therefore, $a\sqrt{2} - a = b\sqrt{2} + b$. Hence $\frac{a}{b} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$.



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Problem 1.5. In the equation $ax^2 + 5x + 2$, solve for a so that the equation has exactly one solution

Solution. First, we know that a quadratic function has only on solution if its determinant is equal to 0. Therefore, $b^2 - 4ac = 0$ and a is equal to $\frac{25}{8}$. However, this is not the only solution because if $a = 0$, the the function becomes linear and has only one solution. So, the values for a are 0 and $\frac{25}{8}$ □

Problem 1.6. Given that $x^2 = 17x + y$ and $y^2 = 17y + x$. Then, what is the value of $\sqrt{x^2 + y^2 + 1}$?

Solution. First, we subtract the equations from each other and get that $(x - y)(x + y) = 16(x - y)$. This simplifies to $x + y = 16$. Now, we add the equations together to get that $x^2 + t^2 = 18(x + y)$, which makes $x^2 + y^2 = 18(16) = 288$. Substituting this in the target equation, we get $\sqrt{x^2 + y^2 + 1} = \sqrt{288 + 1} = 17$ □

Problem 1.7. Given that $\frac{1}{\sqrt[3]{25}+\sqrt[3]{5}+1} = A\sqrt[3]{25} + B\sqrt[3]{5} + C$. So, what is the value of $A + B + C$ (A,B, and C are rational numbers).

Solution. Let x be equal to $\sqrt[3]{5}$. We can, then, write the fraction as $\frac{1}{x^2+x+1}$. We, then, multiply by $\frac{x-1}{x-1}$. We arrive to the following fraction $\frac{x-1}{(x^2+x+1)(x-1)}$, and we can factorize the denominator to be $\frac{x-1}{(x-1)^3}$. This is equal to $\frac{\sqrt[3]{5}-1}{4} = \frac{1}{4}\sqrt[3]{5} - \frac{1}{4}$. Therefore, $A = 0$, $B = \frac{1}{4}$, and $C = \frac{-1}{4}$ □

Problem 1.8. Solve for all real values of x

$$4^x + 6^x = 9^x$$

Solution. First, we divide both sides by 4^x and simplify to get $1 + (\frac{3}{2})^x = (\frac{3}{2})^{2x}$. We, then, let u to be equal to $(\frac{3}{2})^x$. So, we can rewrite the formual as $u^2 - u - 1 = 0$, and since u is positive, we take the positive root of the quadratic function, which is $\frac{1+\sqrt{5}}{2}$. We can now take the natural log of both sides for the equation $u = (\frac{3}{2})^x$. Furthermore, we get $x = \frac{ln(u)}{ln(\frac{3}{2})}$, which simplifies to the value of x to be approximately 1.187. □

Problem 1.9. $sin^2(1°) + sin^2(2°) + sin^2(3°) + \cdots + sin^2(90°)$

Solution. Since $cos^2(x) = sin^2(90 - x)$, then $sin^2(x) + sin^2(90 - x) = 1$. Therefore, all complementary angles are equated to 1. So, we are left with $44 + sin^2(45) + sin^2(90)$ since they have no complementary angle to be summed with. Therefore, the final answer is 45.5 □

Problem 1.10. If $x = \sqrt{3\sqrt{2\sqrt{3\sqrt{2\sqrt{3\sqrt{2...}}}}}}$, Find the value of x^2

Solution. First, since the pattern is infinite, then we can rewrite x to be $x = \sqrt{3\sqrt{2x}}$. After squaring x, we get $x^2 = 3\sqrt{2x}$. After squaring it again, we get that $x^4 = 18x$. Therefore $x^3 = 18$ and $x = \sqrt[3]{18}$. Substituting it in $x^2 = 3\sqrt{2x}$, we get that $x^2 = \sqrt[3]{18^2}$ □

Problem 1.11. Given that $4x^2 + \frac{1}{x^2} = 2$. Then, what is the value of $8x^3 + \frac{1}{x^3}$?

Solution. We realise that the second equation is the sum of two cubes. So, we factorize it to the following form $(2x + \frac{1}{x})(4x^2 + \frac{1}{x^2} - 2)$. And since $4x^2 + \frac{1}{x^2} = 2$, we deduce that $8x^3 + \frac{1}{x^3} = 0$ □

Problem 1.12. The leg of a right-angled triangle is equal to $\frac{1}{5}$ the sum of the other sides, and the triangle's perimeter is 1. What is the area of the triangle?

Solution. Let us say the sides of the triangle are a, b, and c. Therefore, $a+b+c = 1$ and $b = \frac{1}{5}(a+c)$. Substituting this value of b in the first equation, we get the value of b to be $\frac{1}{6}$. Using the Pythagorean theorem, we can substitute with the value of b and substitute with $c^2 = (\frac{5}{6} - a)^2$. We get that $a = \frac{2}{5}$. Therefore, the area of the triangle is $\frac{1}{30}$ □

Problem 1.13. Find all positive integers for which $n^2 + 45$ is a perfect square

Solution. Let $n^2 + 45$ be equal to some perfect square x^2. Therefore, $45 = (x + n)(x - n)$. Looking at the factors of 45, we get 45(1), 15(3), or 9(5). Therefore, we equate the equation to these factors and get three possible values for n, which are 2, 6, and 22 $\square$

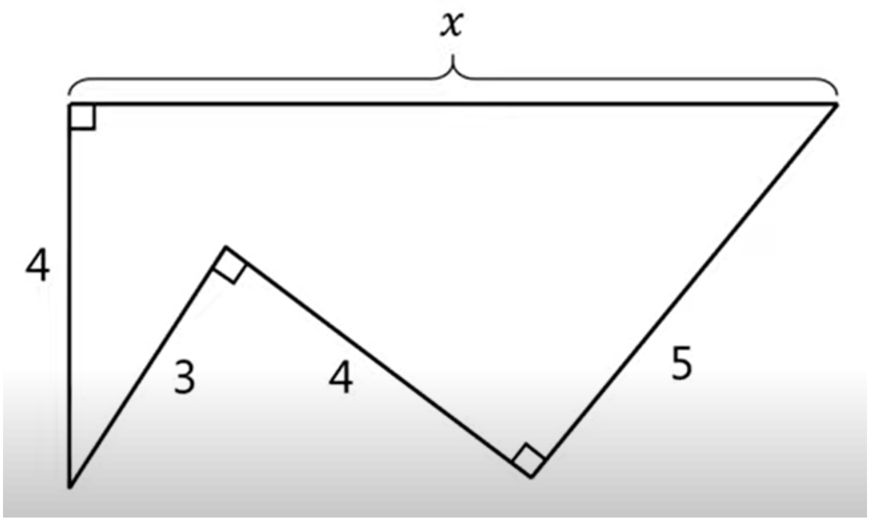
Problem 1.14. A club has 30 members. The positions of president, vice president, and treasurer will be assigned to 3 distinct members. What is the maximum number of distinct assignments that can be made?

Solution. We can use the law for permutations, which is $\frac{n!}{(n!-r!)}$ where n is the number of members and r is the number of people that you will choose. Therefore, the answer is 24360 $\square$

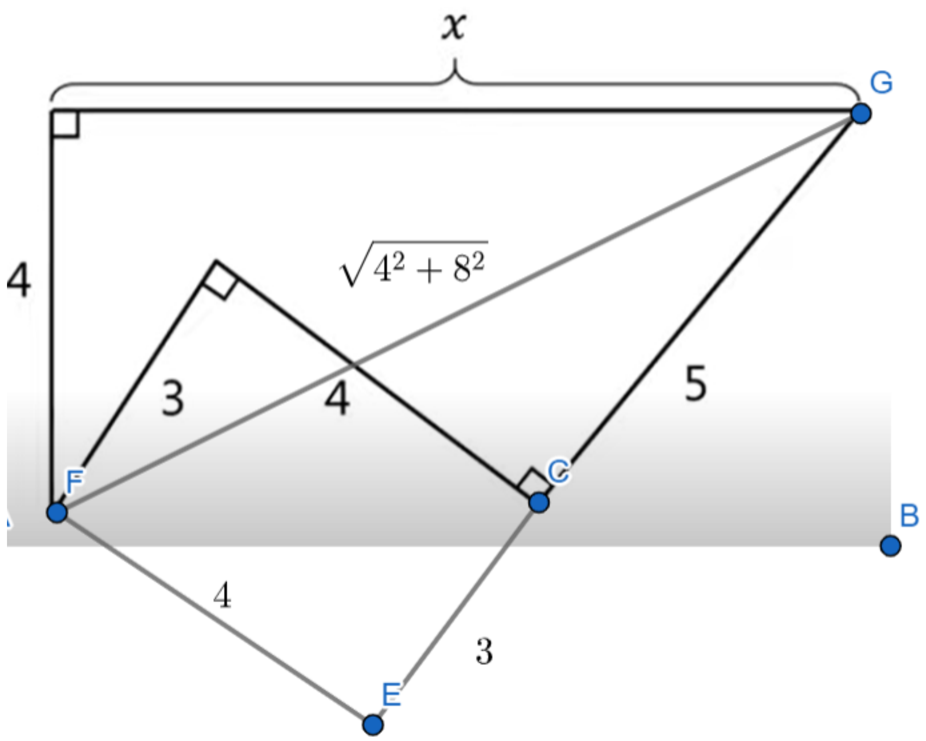
Problem 1.15. There are 90 equally spaced dots marked on a circle. Shannon chooses an integer, n. Beginning at a randomly chosen dot, Shannon goes around the circle clockwise and colours in every nth dot. He continues going around and around the circle colouring in every nth dot, counting each dot whether it is coloured in or not, until he has coloured in every dot. Which of the following could have been Shannon's integer ? $(3, 5, 6, 7)$

Solution. First, we can eliminate any even number since Shanon won't color any odd number in this way. Second, we can also eliminate 3 and 5 since they are divisible by 90, which means he will come to 90 and go over the same path again. Therefore, the only possible choice is 7. $\square$

Problem 1.16. What is the value of x?



Solution. As shown in this figure, we can construct a triangle, where its hypotenuse is $\sqrt{4^2 + 8^2}$. Therefore, $x = 8$



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Problem 1.17. If $x^2 = 2023 + y$, $y^2 = 2023 + x$, where $x \neq y$ and x and y are both real numbers. Find the value of xy

Solution. First, it is clear that

$$x^2 - y = y^2 - x$$

We can rewrite this as

$$(x-y)(x+y+1) = 0$$

This gives the possibilities of either $x = y$ or $x + y = -1$; however, $x = y$ contradicts the givens, which eliminates it. We can substitute x and y with the values

$$x = -1 - y$$

and

$$y = -1 - x$$

We get the following equations

$$x^2 + x - 2022 = 0$$

$$y^2 + y - 2022 = 0$$

We can say that x and y are roots of some function

$$p^2 + p - 2022 = 0$$

Therefore, their product is equal to

$$\frac{-2022}{1} = -2022$$

□

Problem 1.18. Solve for real values of x

$$2\sqrt[3]{2x+1} = x^3 - 1$$

Solution. First, let $\sqrt[3]{2x+1}$ be equal to y. Therefore, we can write y in terms of x to be $y = \frac{x^3-1}{2}$, and we can write x in terms of y to be $\frac{y^3-1}{2}$. These imply that $x = y$, which can be proved geometrically or by contradiction (Hint: Try to see the cases where $x > y$ and $y > x$). Therefore, we can write the cubic function

$$x^3 - 2x - 1 = 0$$

This cubic function has the roots of -1 and $\frac{1\pm\sqrt{5}}{2}$, which is our answer □

Problem 1.19. Simplify and prove your answer for real values of x

$$x = \sqrt[3]{8+3\sqrt{21}} + \sqrt[3]{8-3\sqrt{21}}$$

Solution. First, let $8+3\sqrt{21} = a$ and $8-3\sqrt{21} = b$. Therefore, we can cube both sides of the equation and get

$$x^3 = a + b + 3\sqrt[3]{a^2b} + 3\sqrt[3]{b^2a}$$

We can write the value ab as $(-5)^3$ (since the value ab is the difference between two squares) After substituting and simplifying, we get

$$x^3 = a + b - 15(\sqrt[3]{a}\sqrt[3]{b})$$

Since $x = \sqrt[3]{a}\sqrt[3]{b}$, we can rewrite the equation as a polynomial of the third degree

$$x^3 + 15x - 16 = 0$$

This polynomial has only one real solution which is 1. So, $x = 1$ □

Problem 1.20. Solve for all real solutions of x

$$(x^2 - 7x + 11)^{(x^2-13x+42)} = 1$$

Solution. There are three cases where this equation can yield a solution of 1.

Case 1: The base is equal to 1

$$x^2 - 7x + 11 = 1$$

This equation give us the solutions 2 and 5.

Case 2: The exponent is equal to 0.

$$x^2 - 13x + 42 = 0$$

This equation give us the solutions 6 and 7. (Note: One must also make sure that these solutions don't make the base equal to 0 because this could give the unidentified case of 0^0)

Case 3: The base is equal to -1 and the exponent is an even number.

$$x^2 - 7x + 11 = -1$$

This yields the values 3 and 4 and indeed these values make the exponent even.

Therefore, the solutions for x are $2, 3, 4, 5, 6,$ and 7

□
